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Structure of Vertex Order on Almost Moore Digraphs with no Selfrepeat

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Abstract.

An almost Moore digraph, denoted by (d,k)-digraphs, is a diregular digraph of degree $d > 1$, diameter $k > 1$ and the number of vertices one less than the Moore bound. A vertex v is called repeat u, denoted by $v = r(u)$ if there are two walks of lengths no more than k from u to v. Special case if $u = v$then vertex u is called selfrepeat.The smallest positive integer p such that the composition $r^p(u) = u$ is called order of u. This study focuses on the digraphs with no selfrepeat. Especially, discussing on vertex orders of out-neighbor and in-neighbor of a vertex with smallest order.

Key words: Almost Moore digraph, repeat, selfrepeat, order.

1. Introduction

A topology of network can be modeled by a graph, directed graph or mix graph. Each processing unit can be modeled as vertex, connection between two processing unit as an edge or an arc. In graph theory term, the total number of processing unit on such network is called *order*, the number of connection on each processing unit is called the *degree* of vertex and the maximum comunication time between any two processing unit is called the *diameter* of graph.

Let G be *directed graph* (*digraph*) with natural notation $V(G)$ and $A(G)$ are vertex set and arc set. The natural numbers n, d and k are the number of vertices (order), maximum out degree and diameter of G.One of important problem in graph theory is *degree/diameter* problem for directed graph. The problem is determining the largest order of G when the maximum out-degree and the diameter are given. From the spanning tree, it is easy to see that:

$$n \leq 1 + d + d^2 + \cdots + d^k. \quad (1)$$

The upperbound of Eqs (1) is called *Moore bound* and denoted by $M_{d,k}$. A digraph whose attain the Moore bound is called a *Moore digraph*. The Moore digraphs exist only for few cases, $d = 1$or $k = 1$ [6 or 19].

2. Almost Moore Digraphs

The non existence of Moore digraphs for $d > 1$ and $k > 1$ motivates some researchers to study the existence of almost Moore digraphs,denoted by (d,k)-digraphs, namely digraphs with order $M_{d,k} - 1$.Properties of (d,k)-digraphs have been obtained. One of them isthe diregularity of the digraphs [18].

Let G be a (d,k)-digraph. Since the number of verices one less than the Moore bound then for every vertex u in $V(G)$ there exist exactly a vertex v such that there are two walks of length no more than k from u to v. The vertex v is called *repeatu* and denoted by $v = r(u)$. In case $v = u$, the vertex u is called *selfrepeat*, one of the walk from u to u has length 0 and the other has length k.

The function repeat r is an automorphism on $V(G)$ [4]. For any integer $p \geq 1$, we can define $r^p(u) = r(r^{p-1}(u))$ and $r^0(u) = u$. The smallest integer p such the composition $r^p(u) = u$ is called the order of u. It is clear that a selfrepeat vertex has order 1.

The existence of almost Moore digraph for any degree and diameter do not solved yet. For some cases degree or diameter have been known. Fiol et.al in [13] show that the $(d,2)$-digraphs exist for any degre $d \geq 2$. More spesific Miller and Fris [16] showed that there are exactly three non isomorphic of (2,2)-digraphs. Furthermore Gimbert [15] proved that there is only one (d,2)-digraph for any degree $d \geq 3$.

The non existence of almost Moore digraphs for any degree $d \geq 3$ and diameter $k \geq 3$ still are as open problem. Cholily [7] also Conde et.al [9] have a conjecture that the almost digraphs do not exist for any degree and diameter more than 2. But, at the moment the conjecture still not proved yet. The non existence of almost Moore digraph firstly proved by Miller and Fris in [17] for degree $d = 2$ and diameter $k \geq 3$. For degree $d = 3$, and diameter $k \geq 3$, the $(3,k)$-digraphs also do not exist [5]. Recently, for diameter 3 and 4 have proved that the (d,k)-digraphs do not exist for any degree $d \geq 3$ [11,12].

Some necessary conditions for the existence of almost Moore digraph can be seen in [3,4,14]. For Almost Moore digraph with selfrepeats some neceesary condition have shown in [2,7], in term of order of their vertices.Furthermore, the formula for enumerating of all possible orders of vertices in almost Moore digraphs based on a given repeat structure of out-neighbor of any one selfrepeat vertex can be seen in [1].

3. Almost Moore Digraph with no Selfrepat

Generally, an almost Moore digraph can be classified into two parts namely i) almost Moore digraph with selfrepeat and ii) almost Moore digraph with no selfrepeat. Some necessary conditions about almost Moore digraphs with selfrepeat have mentioned in above. From now on, we assume that the almost Moore digraphs have no selfrepeats.

In this section we discuss some properties of almost Moore digraphs with no selfrepat. More spesific, we discuss about the structure of repeat cycles when we given the smallest order of such digraphs. The proof of the following theorems base on a lemma below was shown in [8].

Let *v* be a vertex in a (*d*,*k*)-digraph *G* for $d \geq 3, k \geq 3$.The set of all out-neighbor vertices of *v* will be denoted by $N^+(v)$. Similarly, the set of all in-neighbor vertices of *v* will be denoted by $N^-(v)$.

Lemma A.*Let* $(v_0, v_1, \dots, v_p)$ *be a walk W of length p in a (d,k)-digraph G. Let m and n (not necessary distinct) be order of* v_0 *and* v_p. *If* $p < k$ *or* ($p \leq k$ *and* $r(v_0) \neq v_p$) *then for each vertex in W has order divide least common multiple of m and n.*

From Lemma A the following Theorems can be proved.

Theorem 1. *Let G be* (*d*,*k*)*-digraph with no selfrepeats. Let p be the smallest vertex order of G and* $p > 2$. *If v is a vertex of G which has order p then* $N^+(v)$ *must contain a vertex of order p.*

Proof. Since $p > 2$ then there exist a vertex *u*, different from *v*, which has order *p* and *u* is not repeat of *v*. Because the diamter of *G* is *K* then there exist a walk from *v* to *u* of length no more than *k*. By Lemma A all verticesin the walk must have order *p*. Hence the assertion follows. □

Let *v* be a vertex of order $p > 2$ in a (*d*,*k*)-digraph *G*. If *p* the smallest order then Theorem 1 shows that the existence of a vertex which have order *p*in out neighbor of *v*. The following theorem gives the structure of relationship between the number of vertices of order *p* in $N^+(v)$ and $N^-(v)$.

Theorem 2.*Let G be a* (*d*,*k*)*-digraph contain no selfrepeat and p the smallest vertex order of G*, $p > 2$. *If v is a vertex of order p in G then the number of vertices of order p in* $N^+(v)$ *and* $N^-(v)$ *are the same.*

Proof. Let$N^+(v) = \{v_1, v_2, \dots, v_d\}$. The existence of a vertex of order *p* in $N^+(v)$ is guaranted by Theorem 1. If all of vertices in $N^+(v)$have oder p then by using Lemma A, all vertices in $N^-(v)$also have have order *p*. This can be seen from di disjoint walks of no more than *k* from each v_i, $i \in \{1, 2, \dots, d\}$, to $r(v)$.Since $N^-(r(v)) = r(N^-(v))$ then all vertices of $N^-(v)$ have order *p*.

Let $v_1, v_2, \dots, v_l$, $1 \leq l \leq d - 1$ be vertices of order *p* in $N^+(v)$. For each $i \in \{1, 2, \dots, l\}$ there exists a path of length less than or equal to *k* from v_i to $r(v)$. These *l* paths have only the vertex $r(v)$ in common. Since the order *v* and $r(v)$ are the same then by Lemma A, the set $N^-(r(v))$ contains at least *l* vertices of order *p*. Since $N^-(r(v)) = r(N^-(v))$ then there exist at least *l* vertices of $r(N^-(v))$ which have order *p*. Moreover, $N^-(v)$ contains at least *l* vertices of order *p*.

Assume there exist more than *l* vertices of order *p* in $N^-(v)$. Then, $N^+(v)$ also contains more than *l* vertices of order *p*. This is contradiction. Thus the assertion follows. □

4. Open problem

Existence problem of (d,k)-digraphs for any degree and diameter still are open. Some researchers study about the existence and some others study about the construction technique. Since the study of the existence still very hard, then some researcher study the necessary condition of this existence. Since there are two types of almost Moore digraphs namely contain selfrepeat or none then we propose some open problems for almost Moore digraphs without selfrepeat.

Open problem 1.
How to enumarte the vertex order almost Moore digraphs with no selfrepeat?

Open Problem 2.
If the smallest vertex order of almost Moore digraphs is$p$, is there a vertex of order other than p or $2p$?

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